PART 1: QUESTIONS

Name:______________________________ **Age:**_____ **Id:**_______________ **Course:**__________

Trigonometry - Exam 2 **Lesson: 4-7**

Instructions:

- Please begin by printing your Name, your Age, your Student Id , and your Course Name in the box above and in the box on the solution sheet.
- You have 90 minutes (class period) for this exam.
- You can not use any calculator, computer, cellphone, or other assistance device on this exam. However, you can set our flag to ask permission to consult your own one two-sided-sheet notes at any point during the exam (You can write concepts, formulas, properties, and procedures, but questions and their solutions from books or previous exams are not allowed in your notes).
- Each multiple-choice question is worth 5 points and each extra essay-question is worth from 0 to 5 points. (Even a simple related formula can worth some points).
- Set up your flag if you have a question.
- Relax and use strategies to improve your performance.

Exam Strategies to get the best performance:

- Spend 5 minutes reading your exam. Use this time to classify each Question in (E) Easy, (M) Medium, and (D) Difficult.
- Be confident by solving the easy questions first then the medium questions.
- Be sure to check each solution. In average, you only need 30 seconds to test it. (Use good sense).
- Don't waste too much time on a question even if you know how to solve it. Instead, skip the question and put a circle around the problem number to work on it later. In average, the easy and medium questions take up half of the exam time.
- Solving the all of the easy and medium question will already guarantee a minimum grade. Now, you are much more confident and motivated to solve the difficult or skipped questions.
- Be patient and try not to leave the exam early. Use the remaining time to double check your solutions.

1. Given $\cos(a+b) = \cos a \cos b - \sin a \sin b$, then:

a) $\cos 2x = 2\cos^2 x + 1$
b) $\cos 2x = \cos^2 x + \sin^2 x$
c) $\cos 2x = \sin^2 x - \cos^2 x$
d) $\cos 2x = 2\cos^2 x - 1$
e) None of the above.

Solution: d

Since $\cos(a+b) = \cos a \cos b - \sin a \sin b$
For $a = x$ and $b = x$, we have:
$\cos(x+x) = \cos x \cos x - \sin x \sin x$
$\cos 2x = \cos^2 x - \sin^2 x$
Since $\sin^2 x + \cos^2 x = 1$ then
$\cos 2x = \cos^2 x - (1 - \cos^2 x)$
Thus, $\cos 2x = 2\cos^2 x - 1$.

2. Simplify $\sin(x - \pi)$:
a) $\sin x$
b) $\cos x$
c) $-\cos x$
d) $-\sin x$
e) None of the above.

Solution: d

Special Case of Addition Formulas

There is a special case when the angle a or b is on the horizontal or vertical axes:

Rule1: If there is a vertical angle in the addition formula the function changes to its cofunction. If there is a horizontal angle in the addition formula then the function does not change.

Rule 2: The sign of the original function is the same as the sign in front of the new function.

Thus, $\sin(x - \pi) = -\sin x$.

3. Simplify $\cot(x + \pi)$:
a) $\tan x$
b) $-\cot x$
c) $-\tan x$
d) $\cot x$
e) None of the above.

Solution: d

Special Case of Addition Formulas

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Rule 2: The sign of the original function is the same as the sign in front of the new function.

Thus, $\cot(x + \pi) = \cot x$.

4. Simplify $\cos(x - \frac{3\pi}{2})$:
a) $\sin x$
b) $\cos x$
c) $-\cos x$
d) $-\sin x$
e) None of the above.

Solution: d

Special Case of Addition Formulas

There is a special case when the angle a or b is on the horizontal or vertical axes:

Rule1: If there is a vertical angle in the addition formula the function changes to its cofunction. If there is a horizontal angle in the addition formula then the function does not change.

Rule 2: The sign of the original function is the same as the sign in front of the new function.

Thus, $\cos(x - \frac{3\pi}{2}) = -\sin x$.

5. Simplify $\cot\left(x + \frac{3\pi}{2}\right)$:
a) $\tan x$
b) $\cot x$
c) $-\cos x$
d) $-\cot x$
e) None of the above.

Solution: e

Special Case of Addition Formulas

There is a special case when the angle a or b is on the horizontal or vertical axes:

Rule1: If there is a vertical angle in the addition formula the function changes to its cofunction. If there is a horizontal angle in the addition formula then the function does not change.

Rule 2: The sign of the original function is the same as the sign in front of the new function.

Thus, $\cot\left(x + \frac{3\pi}{2}\right) = -\tan x$.

6. The value of $\sin 75^\circ$ is

a) $\frac{\sqrt{2}}{4}(\sqrt{3}+1)$
b) $\frac{\sqrt{2}}{4}(\sqrt{3}-1)$
c) $-\frac{\sqrt{2}}{4}(\sqrt{3}-1)$
d) $-\frac{\sqrt{2}}{4}(\sqrt{3}+1)$
e) None of the above.

Solution: a

$\sin 75^\circ = \sin\left(45^\circ + 30^\circ\right)$
$\sin 75^\circ = \sin 45^\circ \cos 30^\circ + \sin 30^\circ \cos 45^\circ$
$\sin 75^\circ = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2}$
$\sin 75^\circ = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$.

7. The value of $\cos 15^\circ$ is

a) $\frac{\sqrt{2}}{4}(\sqrt{3}-1)$
b) $\frac{\sqrt{2}}{4}(\sqrt{3}+1)$
c) $-\frac{\sqrt{2}}{4}(\sqrt{3}-1)$
d) $-\frac{\sqrt{2}}{4}(\sqrt{3}+1)$
e) None of the above.

Solution: b

$\cos\left(15^\circ\right) = \cos\left(45^\circ - 30^\circ\right)$
$\cos 15^\circ = \cos 45^\circ \cos 30^\circ + \sin 30^\circ \sin 45^\circ$
$\cos 15^\circ = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2}$
$\cos 15^\circ = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$.

8. The value of $\tan(-105^\circ)$ is

a) $3+\sqrt{2}$
b) $3-\sqrt{2}$
c) $-2+\sqrt{3}$
d) $-2-\sqrt{3}$
e) None of the above.

Solution: c

First, let go to find $\tan 105^\circ$ because $\tan(-105^\circ) = -\tan 105^\circ$ (Easy to check in the trigonometric ball).

$\tan\left(105^\circ\right) = \tan\left(45^\circ + 60^\circ\right)$
$\tan 105^\circ = \frac{\tan 45^\circ + \tan 60^\circ}{1 - \tan 45^\circ \tan 60^\circ}$
$\tan 105^\circ = \frac{1 + (\sqrt{3})}{1 - 1(\sqrt{3})}$
$\tan 105^\circ = \frac{1+\sqrt{3}}{1-\sqrt{3}}$
$\tan 105^\circ = \frac{(1+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}$
$\tan 105^\circ = -2-\sqrt{3}$

Thus, $\tan(-105^\circ) = 2+\sqrt{3}$.

9. The value of $\cos 2x - \cos^2 x$ is:

a) $1 + \sin 2x$
b) $1 - \sin 2x$
c) $\cos^2 x$
d) $\sin^2 x$
e) None of the above.

Solution: e

$\cos 2x - \cos^2 x = \cos^2 x - \sin^2 x - \cos^2 x$
$\cos 2x - \cos^2 x = -\sin^2 x.$

10. Solve $\sin 2x + \cos x = 0$

a) $\left.\begin{array}{l} x = \frac{\pi}{2} + \pi k \\ x = \frac{\pi}{6} + 2\pi k \\ x = \frac{5\pi}{6} + 2\pi k \end{array}\right\} \quad k \in \mathbb{Z}$

b) $\left.\begin{array}{l} x = \frac{\pi}{2} + \pi k \\ x = \frac{7\pi}{6} + 2\pi k \\ x = \frac{11\pi}{6} + 2\pi k \end{array}\right\} \quad k \in \mathbb{Z}$

c) $\left.\begin{array}{l} x = \pi k \\ x = \frac{2\pi}{3} + 2\pi k \\ x = \frac{4\pi}{3} + 2\pi k \end{array}\right\} \quad k \in \mathbb{Z}$

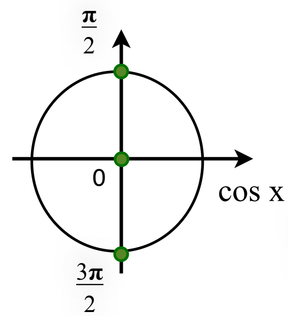
d) $\left.\begin{array}{l} x = \pi k \\ x = \frac{\pi}{3} + 2\pi k \\ x = \frac{5\pi}{3} + 2\pi k \end{array}\right\} \quad k \in \mathbb{Z}$

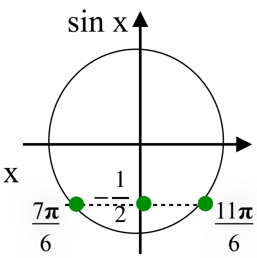
e) None of the above.

Solution: b

$\sin 2x + \cos x = 0$
$2 \sin x \cos x + \cos x = 0$
$\cos x (2 \sin x + 1) = 0$

$\cos x = 0$ or $\sin x = -\frac{1}{2}$





$\left.\begin{array}{l} x = \frac{\pi}{2} + \pi k \\ x = \frac{7\pi}{6} + 2\pi k \\ x = \frac{11\pi}{6} + 2\pi k \end{array}\right\} \quad k \in \mathbb{Z}$

11. Solve: $\sin(-3x) - \sin(x) = 0$

a) $x = \frac{\pi k}{4}, k \in \mathbb{Z}$

b) $x = \frac{\pi}{4} + \frac{\pi k}{2}$ or $x = \pi k, k \in \mathbb{Z}$

c) $x = \frac{\pi}{4} + \frac{\pi k}{2}$ and $x = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$

d) $x = \frac{\pi}{2} + \pi k$ and $x = \pi k, k \in \mathbb{Z}$

e) None of the above.

Solution: e

$\sin(-3x) - \sin(x) = 0$

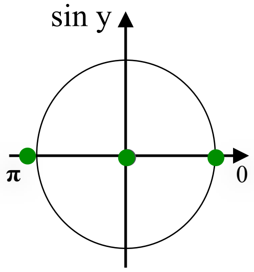
$2 \sin\left(\frac{(-3x) - x}{2}\right) \cos\left(\frac{(-3x) + x}{2}\right) = 0$

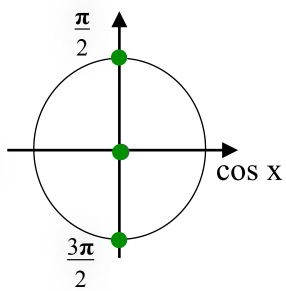
$2 \sin(-2x)\cos(-x) = 0$

Since $2 \sin(-2x) = -2 \sin(2x)$ and $\cos(-x) = \cos(x)$, then:

$-2 \sin(2x)\cos(x) = 0$

$\sin 2x = 0$ or $\cos x = 0$
$y = 2x$
$\sin y = 0$





$y = \pi k$ $\quad x = \dfrac{\pi}{2} + \pi k, k \in \mathbb{Z}$

$x = \dfrac{\pi k}{2}, k \in \mathbb{Z}$

Thus, $x = \dfrac{\pi k}{2}, k \in \mathbb{Z}$.

12. Solve: $\cos 3x - \cos x = 0$

a) $x = \dfrac{\pi k}{2}, k \in \mathbb{Z}$

b) $x = \dfrac{\pi}{4} + \dfrac{\pi k}{2}$ or $x = \pi k, k \in \mathbb{Z}$

c) $x = \dfrac{\pi}{4} + \dfrac{\pi k}{2}$ and $x = \dfrac{\pi}{2} + \pi k, k \in \mathbb{Z}$

d) $x = \dfrac{\pi}{2} + \pi k$ and $x = \pi k, k \in \mathbb{Z}$

e) None of the above.

Solution: a

$\cos 3x - \cos x = 0$

$$-2\sin\left(\frac{3x + x}{2}\right)\sin\left(\frac{3x - x}{2}\right) = 0$$

$-2\sin 2x \sin x = 0$

$\sin 2x = 0$ or $\sin x = 0$
$y = 2x$
$\sin y = 0$

$y = \pi k$ $\quad x = \pi k, k \in \mathbb{Z}$

$x = \dfrac{\pi k}{2}, k \in \mathbb{Z}$

Thus, $x = \dfrac{\pi k}{2}, k \in \mathbb{Z}$.

13. Solve: $\tan(-3x) + \tan x = 0$

a) $x = \dfrac{\pi}{2} + \pi k, k \in \mathbb{Z}$

b) There is no solution.

c) $x = \pi k, k \in \mathbb{Z}$

d) $x = \dfrac{\pi k}{2}, k \in \mathbb{Z}$

e) None of the above.

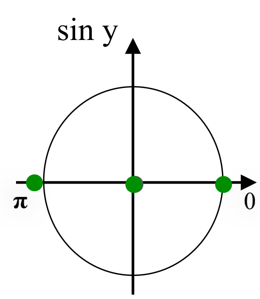
Solution: b

$\tan(-3x) + \tan x = 0$

$$\tan(-3x) + \tan x = \frac{\sin(-3x + x)}{\cos(-3x)\cos x}$$

Existence condition: $\cos(-3x)\cos x \neq 0$
$\left(x \neq \dfrac{\pi}{6} + \dfrac{\pi k}{3} \text{ or } x \neq \pi k\right)$

$\sin(-2x) = 0$
$-\sin 2x = 0$
$\sin 2x = 0$
$y = 2x$
$\sin y = 0$

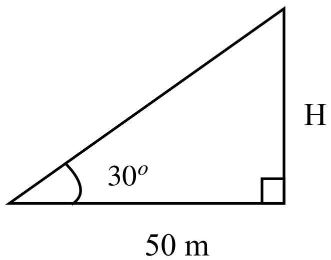


$y = \pi k$
$2x = \pi k$
$x = \dfrac{\pi k}{2}, k \in \mathbb{Z}$. Then by the existence condition,
$s = \emptyset$.

14. A person 50 meters from the base of a tree, observes that the angle between the ground and the top of the tree is 30 degrees. Estimate the height H (m) of the tree.

a) $\frac{10\sqrt{3}}{3}$ b) $\frac{20\sqrt{3}}{3}$ c) $10\sqrt{3}$ d) $\frac{50\sqrt{3}}{3}$ e) $20\sqrt{3}$

Solution: d



$\tan(30^o) = \frac{H}{50} \Rightarrow \frac{\sqrt{3}}{3} = \frac{H}{50} \Rightarrow H = \frac{50\sqrt{3}}{3}m$

15. Find $\sin(3x)$.

a) $4\sin x - 3\sin^3 x$
b) $3\sin x - 4\sin^3 x$
c) $3\sin x - 4\sin^2 x$
d) $4\sin x - 3\sin^2 x$
e) None of the above.

Solution: b

$$\begin{aligned}\sin(3x) &= \sin(x+2x)\\ &= \sin x\cos(2x) + \sin(2x)\cos x\\ &= \sin x(1-2\sin^2 x) + (2\sin x\cos x)\cos x\\ &= \sin x(1-2\sin^2 x) + 2\sin x(1-\sin^2 x)\\ &= \sin x - 2\sin^3 x + 2\sin x - 2\sin^3 x\\ &= 3\sin x - 4\sin^3 x\end{aligned}$$

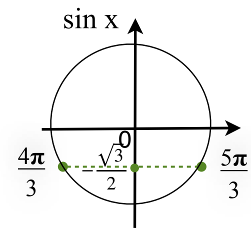
16. Find the value of $\theta = \arcsin\left(-\frac{\sqrt{3}}{2}\right)$.

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{5\pi}{3}$
d) $\theta = \frac{7\pi}{4}$
e) None of the above.

Solution: c

$\theta = \arcsin\left(-\frac{\sqrt{3}}{2}\right)$

$\sin\theta = -\frac{\sqrt{3}}{2}$



$\theta = \frac{4\pi}{3}$ Discarded $x \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\theta = \frac{5\pi}{3}$ OK $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

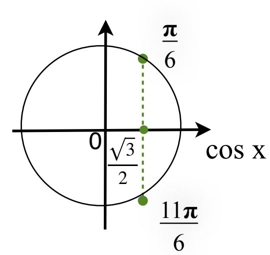
17. Find the value of $\theta = \arccos\left(\frac{\sqrt{3}}{2}\right)$.

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{5\pi}{6}$
d) $\theta = \frac{\pi}{6}$
e) None of the above.

Solution: d

$\theta = \arccos\left(\frac{\sqrt{3}}{2}\right)$

$\cos\theta = \frac{\sqrt{3}}{2}$



$\theta = \frac{\pi}{6}$ OK $x \in [0,\pi]$

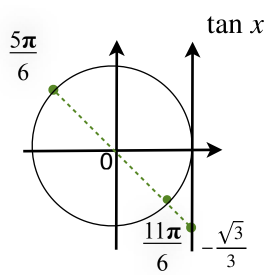
$\theta = \frac{11\pi}{6}$ Discarded $x \notin [0,\pi]$.

18. Find the value of $\theta = \arctan\left(-\dfrac{\sqrt{3}}{3}\right)$.

a) $\theta = \dfrac{\pi}{4}$
b) $\theta = \dfrac{\pi}{3}$
c) $\theta = \dfrac{5\pi}{3}$
d) $\theta = \dfrac{11\pi}{6}$
e) None of the above.

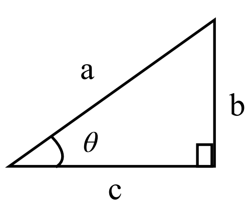
Solution: d

$\theta = \arctan\left(-\dfrac{\sqrt{3}}{3}\right)$
$\tan\theta = -\dfrac{\sqrt{3}}{3}$



$\theta = \dfrac{5\pi}{6}$ Discarded $x \notin \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$
$\theta = \dfrac{11\pi}{6}$ OK $x \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$.

19. Find $\cos(2\theta)$, where $0 < \theta < \dfrac{\pi}{2}$:



a) $\dfrac{2bc}{a^2}$
b) $\dfrac{c^2 - b^2}{a^2}$
c) $\dfrac{2bc}{c^2 - b^2}$
d) $\dfrac{a^2}{c^2 - b^2}$
e) None of the above.

Solution: b

$\sin\theta = \dfrac{b}{a}$ and $\cos\theta = \dfrac{c}{a}$ then:
$\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$\cos(2\theta) = \left(\dfrac{c}{a}\right)^2 - \left(\dfrac{b}{a}\right)^2$
$\cos(2\theta) = \dfrac{c^2 - b^2}{a^2}$.

20. Given $t = \tan\left(\dfrac{x}{2}\right)$ then:

I. $\sin x = \dfrac{2t^2}{1 + t^2}$
II. $\cos x = \dfrac{1 - t^2}{1 + t^2}$
III. $\tan x = \dfrac{2t}{1 - t^2}$, where $t \neq \pm 1$.

Then:

a) Only I and II are correct
b) Only I and III are correct
c) Only II and III are correct
d) I, II, and III are correct
e) None of the above.

Solution: c

I. False

$\sin x = \dfrac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$

$\sin x = \dfrac{2t}{1 + t^2}$

II. True

$\cos x = \dfrac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$

$\cos x = \dfrac{1 - t^2}{1 + t^2}$

III. True

$$\tan x = \frac{\sin x}{\cos x}$$

$$\tan x = \frac{\left(\frac{2t}{1+t^2}\right)}{\left(\frac{1-t^2}{1+t^2}\right)}$$

$$\tan x = \frac{2t}{1-t^2}, \text{ where } t \neq \pm 1.$$

PART 2: SOLUTIONS

Consulting

Name:______________________________ **Age:**_____ **Id:**_______________ **Course:**__________

Multiple-Choice Answers

Questions	A	B	C	D	E
1					
2					
3					
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	Points	Max
Multiple Choice		100
Extra Points		25
Consulting		10
Age Points		25
Total Performance		160
Grade		A

Extra Questions

21. Solve: $\cos^{-1}\left[\sin\left(\frac{\pi}{13}\right)\right]$,

where $\cos^{-1} x$ is the inverse function of $\cos x$.

Solution:

Since $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.

$$\sin\left(\frac{\pi}{13}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{13}\right) = \cos\left(\frac{13\pi - 2\pi}{26}\right) = \cos\left(\frac{11\pi}{26}\right)$$

Thus,

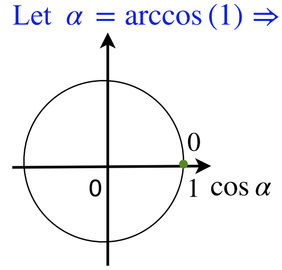
$$\cos^{-1}\left[\sin\left(\frac{\pi}{13}\right)\right] = \cos^{-1}\left[\cos\left(\frac{11\pi}{26}\right)\right] = \frac{11\pi}{26}.$$

22. Find the value of the expression:

$$\tan\left[\arccos(1) - \arcsin(1) + \arctan(-1)\right]$$

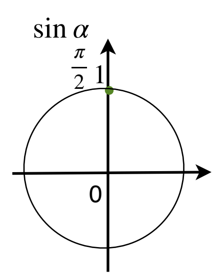
Solution: d

Let $\alpha = \arccos(1) \Rightarrow \cos(\alpha) = 1$



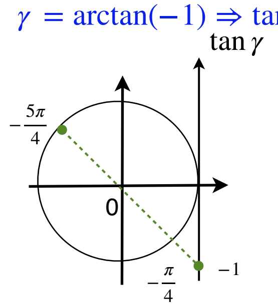
$\alpha = 0$ OK $\alpha \in [0,\pi]$

$\beta = \arcsin(1) \Rightarrow \sin(\beta) = 1$



$\beta = \frac{\pi}{2}$ OK $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\gamma = \arctan(-1) \Rightarrow \tan(\gamma) = -1$



$\gamma = -\frac{\pi}{4}$ OK $\alpha \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$\gamma = -\frac{5\pi}{4}$ Discarded $\alpha \notin \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$\tan(\alpha - \beta + \gamma) = \tan\left(0 - \frac{\pi}{2} - \frac{\pi}{4}\right) = \tan\left(-\frac{3\pi}{4}\right) = 1$

Thus, $\tan\left[\arccos(1) - \arcsin(1) + \arctan(-1)\right] = 1.$

23. Solve:

$\sin(2x) = \sin x$; where $0 \leq x < 2\pi$

Solution:

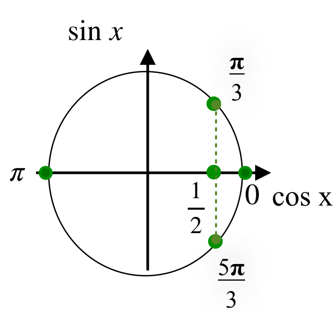
$\sin(2x) = \sin x$; where $0 \leq x < 2\pi$

$2\sin x \cos x = \sin x$

$2\sin x \cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

$\sin x = 0$ or $\cos x = \frac{1}{2}$



$S = \left\{0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$

24. Simplify: $\frac{\sin(2x)}{\sin x \cos x}$

Solution:

$\frac{\sin(2x)}{\sin x \cos x} = \frac{2\sin x \cos x}{\sin x \cos x} = 2.$

25. Solve: $-\cos x - \sin x = -1$, where $x \in [0,2\pi]$

Hint: There are severals method to solve it including:

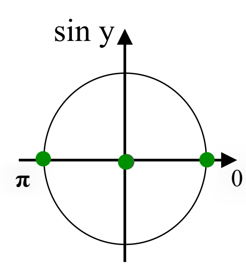
1. Squaring both side of the equation
2. Addition formulas
3. Sum to Product formulas
4. Tangent Half-angle formulas.

Choose any method to receive a full credit of 10 points, but I will give you additional 5 points for each correct extra additional method. (Maximum question value is 25 points).

Solutions:

1. Squaring both side of the equation

$-\cos x - \sin x = -1$
$\cos x + \sin x = 1$
$(\sin x + \cos x)^2 = (1)^2$
$\sin x^2 + 2\sin x \cos x + \cos^2 x = 1$
Since $\sin^2 x + \cos^2 x = 1$ then:
$2\sin x \cos x = 0$
$2\sin(2x) = 0$
Let $y = 2x$ then:
$\sin y = 0$



$y = \pi k$, where $k \in \mathbb{Z}$
$2x = \pi k$
$x = \frac{\pi k}{2}$, where $k \in \mathbb{Z}$
$k = 0 \Rightarrow x = 0$ OK

$k = 1 \Rightarrow x = \frac{\pi}{2}$ OK

$k = 2 \Rightarrow x = \pi$ Discarded

$k = 3 \Rightarrow x = \frac{3\pi}{2}$ Discarded

$k = 4 \Rightarrow x = 2\pi$ OK

Thus, $S = \left\{0, \frac{\pi}{2}, 2\pi\right\}$

2. Addition formulas

$-\cos x - \sin x = -1$

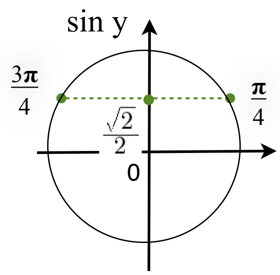
$\cos x + \sin x = 1$

$\left(\frac{\sqrt{2}}{2}\right)\cos x + \sin x\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$

$\sin\left(\frac{\pi}{4} + x\right) = \frac{\sqrt{2}}{2}$

$\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Let $y = x + \frac{\pi}{4} \Rightarrow \sin y = \frac{\sqrt{2}}{2}$



$y = \frac{\pi}{4} + 2\pi k$ or $y = \frac{3\pi}{4} + 2\pi k$, where $k \in \mathbb{Z}$

$x - \frac{\pi}{4} = \frac{\pi}{4} + 2\pi k$ $\quad x - \frac{\pi}{4} = \frac{3\pi}{4} + 2\pi k$

$x = \frac{\pi}{2} + 2\pi k$ $\quad x = \pi + 2\pi k$, where $k \in \mathbb{Z}$

Thus, $S = \left\{0, \frac{\pi}{2}, 2\pi\right\}$

3. Sum to Product formulas

$-\cos x - \sin x = -1$

$\cos x + \sin x = 1$

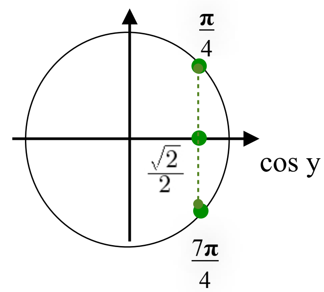
$\sin\left(\frac{\pi}{2} - x\right) + \sin x = 1$

$2\sin\left(\frac{\left(\frac{\pi}{2} - x\right) + x}{2}\right)\cos\left(\frac{\left(\frac{\pi}{2} - x\right) - x}{2}\right) = 1$

$2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4} - x\right) = 1$

$\cos\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Let $y = x - \frac{\pi}{4} \Rightarrow \cos y = \frac{\sqrt{2}}{2}$



$y = \frac{\pi}{4} + 2\pi k$ or $y = \frac{7\pi}{4} + 2\pi k$, where $k \in \mathbb{Z}$

$x - \frac{\pi}{4} = \frac{\pi}{4} + 2\pi k$ $\quad x - \frac{\pi}{4} = \frac{7\pi}{4} + 2\pi k$

$x = \frac{\pi}{2} + 2\pi k$ $\quad x = 2\pi + 2\pi k$, where $k \in \mathbb{Z}$

Thus, $S = \left\{0, \frac{\pi}{2}, 2\pi\right\}$

4. Tangent Half-angle formulas

$-\cos x - \sin x = -1$

$\cos x + \sin x = 1$

Let $t = \tan\left(\frac{x}{2}\right)$. Then:

$\sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$ $\quad \cos x = \frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$

$\sin x = \frac{2t}{1 + t^2}$ $\quad \cos x = \frac{1 - t^2}{1 + t^2}$

$\frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2} = 1$

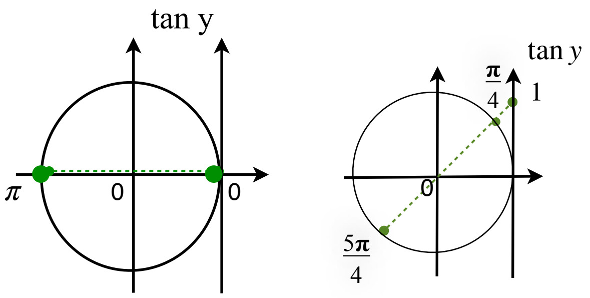
$\frac{1 - t^2 + 2t}{1 + t^2} = 1$

$1 - t^2 + 2t = 1 + t^2$
$2t^2 - 2t = 0$
$2t(t - 1) = 0$

$t = 0$ or $t = 1$

$\tan\left(\frac{x}{2}\right) = 0$ $\tan\left(\frac{x}{2}\right) = 1$

Let $y = \frac{x}{2}$ then:



$y = \pi k$ $y = \frac{\pi}{4} + \pi k$, where $k \in \mathbb{Z}$

$\frac{x}{2} = \pi k$ $\frac{x}{2} = \frac{\pi}{4} + \pi k$

$x = 2\pi k$ $x = \frac{\pi}{2} + 2\pi k$, where $k \in \mathbb{Z}$

Thus, $S = \left\{0, \frac{\pi}{2}, 2\pi\right\}$